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Constructive characterizations of bar subsets

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Abstract

We provide some constructive characterizations of the notion of *bar subset* for the complete binary tree, alias Cantor space, for the complete countable spreading tree, alias Baire Space, and, more generally, for an inductively generated formal topology. Moreover, by using a completeness theorem for inductively generated formal topologies, we prove that such characterizations are classically equivalent to the standard one.

Keywords: Constructive mathematics; Bar theorem; Formal topology

1. Bar subsets

Let us begin by recalling informally the standard notion of bar subset for the collection of the infinite sequences $\alpha, \beta, \ldots$ of boolean values (respectively, for the infinite sequences of natural numbers). As usual, we write $\overline{\alpha}(n)$ to mean the finite sequence $\alpha(0), \ldots, \alpha(n-1)$ of the first n values of the infinite sequence α. Now, let V be a set of finite sequences. Then V is a *bar* if and only if for all infinite sequences α there exists a natural number n such that $\overline{\alpha}(n) \in \overline{V}$, where $\overline{V}$ is the subset of the finite sequences extending some finite sequence in V.

The notion of bar subset is problematic from a constructive viewpoint since it requires one to understand the meaning of the universal quantification on the *collection* of the infinite sequences while universal quantification has a constructive meaning only on *sets* (see [7] for a discussion of the distinctions between sets, which have to be inductively generated, and collections).

There is however an inductive, and hence constructive, approach to the definition of bar subset which is based on a sort of simple deduction system (see [6] or just wait for the next section). Then, one may wonder whether this inductive definition is equivalent to the standard one, namely, one can be interested in proving what is called Brouwer's fan theorem in the case of the complete binary tree or the bar theorem in the case of the complete countable spreading tree. However, they seem not to admit a constructive proof (see for instance [4]) and also our proof, that can be found in Section 1.2, is only classical.

The suggestion is then to take the inductive definition as the rigorous definition of what it means for a subset of finite sequences to be a bar and use such a definition as an explanation of the quantification over all infinite sequences (see the introduction of [6]). However, if one follows this approach, it is useful to understand what the meaning of the

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inductive definition is. To this end we will provide many alternative characterizations for the bar subsets and prove their equivalence with the inductive one in a fully constructive and algorithmic way.

1.1. Inductive bar subsets

In order to express in a completely formal way the statements of the fan and the bar theorems we recall the definitions of some sets in Martin-Löf type theory that we choose here as our constructive framework (see [7]). First of all, the set of boolean values is defined by using the following inductive rules:

$$(\mathsf{Boole}\text{-introduction}) \quad 0 \in \mathsf{Boole} \qquad 1 \in \mathsf{Boole}.$$

Then, the set of the list of boolean values can be defined as follows:

$$(\mathsf{ListB}\text{-introduction}) \quad \mathsf{nil} \in \mathsf{ListB} \qquad \frac{a \in \mathsf{Boole} \quad l \in \mathsf{ListB}}{la \in \mathsf{ListB}}.$$

One can easily define a function $\mathsf{length}(-)$ which yields the length of a list and it is only a bit more complex to define an order relation between elements of ListB which states that $x \sqsubseteq y$ if and only if y is an initial segment of x.

The set ListB of the lists of boolean values can be identified with the complete binary tree provided the root of the tree is identified with the empty list nil and every other point of the complete binary tree is identified with the list which codes the path from the root to such a point; hence a subset of points of the binary tree can be identified with a subset of lists.

Thus, we are ready to introduce the notion of inductive bar subset. To this end, let $l \in \mathsf{ListB}$ and $U \subseteq \mathsf{ListB}$; then we will write $l \lhd U$ to mean that any continuation of the list l is barred by U. Hence, it is easy to see that the following conditions are valid:

$$(\text{extended reflexivity}) \ \frac{l \in \overline{U}}{l \lhd U} \qquad (\text{infinity}) \ \frac{l0 \lhd U \quad l1 \lhd U}{l \lhd U}$$

where $\overline{U} \equiv \{l \in \mathsf{ListB} \mid (\exists m \in U)\ l \sqsubseteq m\}$.

Now, provided we use *extended reflexivity* and *infinity* as the rules for an inductive definition of the proposition $- \lhd -$, we immediately arrive at the following definition which formalizes the idea that the subset U is a bar when any path in the complete binary tree is barred by U.

Definition 1.1 (*Inductive Bar Subset*). Let U be a subset of the complete binary tree. Then U is an *inductive bar subset* if and only if $\mathsf{nil} \lhd U$.

What has been done above can be generalized to the notion of a *spread* (see [13,4]), namely, a complete countable spreading tree, if we use the set of the lists over natural numbers instead of the set of the lists over boolean values. To this end, let us recall that the set of natural numbers is defined by using the following inductive rules:

$$(\mathsf{N}\text{-introduction}) \quad 0 \in \mathsf{N} \qquad \frac{a \in \mathsf{N}}{a+1 \in \mathsf{N}}$$

and the set of the lists of natural numbers is obtained by using the following rules:

$$(\mathsf{ListN}\text{-introduction}) \quad \mathsf{nil} \in \mathsf{ListN} \qquad \frac{a \in \mathsf{N} \quad l \in \mathsf{ListN}}{la \in \mathsf{ListN}}.$$

Now, it is not difficult to adapt the conditions proposed above for the case of the complete binary tree to such a new setting:

$$(\text{extended reflexivity}) \ \frac{l \in \overline{U}}{l \lhd U} \qquad (\text{infinity}) \ \frac{(\forall x \in \mathsf{N})\ lx \lhd U}{l \lhd U}.$$

Then, to state what is an inductive bar subset for a spread we can use a definition obtained by Definition 1.1 by just replacing the words "complete binary tree" with "complete countable spreading tree".

It is interesting to note that *extended reflexivity* and *infinity* are exactly the rules which define the cover relation of an inductively generated formal topology on the set ListB or the set ListN. In fact, the cover relation of an *inductively generated formal topology* $\mathcal{A}$ on a set A, ordered by a reflexive and transitive relation $\leq$, with an axiom set constituted by an indexed family of sets $I(a)$ set, for any $a \in A$, and a family of subsets $C(a, i) \subseteq A$, for any $a \in A$ and $i \in I(a)$, is the infinitary relation between elements and subsets of A inductively generated by using the following rules:

$$\text{(extended reflexivity)} \quad \frac{a \in \overline{U}}{a \lhd U}$$

$$\text{(infinity)} \quad \frac{i \in I(a) \quad (\forall y \in C(a, i))\ y \lhd U}{a \lhd U}$$

where $\overline{U} \equiv \{x \in A \mid (\exists u \in U)\ x \leq u\}$.

The only requirement is that the axiom set $I(-)$, $C(-, -)$ is *localized*, namely, if $a \leq c$ and $i \in I(c)$ then there exists an index $j \in I(a)$ such that $C(a, j) \subseteq \{x \in A \mid (x \leq a)\ \&\ (\exists y \in C(c, i))\ x \leq y\}$,

The intended meaning of the cover relation $a \lhd U$ is describing the topological situation where a is a basic open, U is a set of basic opens and a is a subset of the union of the basic opens in U (see [3] or [15] in order to have a full presentation of inductively generated formal covers and their properties,[1] but also the last sections of [11] where the connection between formal covers and the bar theorem is analyzed).

Now, it is not difficult to see that in the case of the formal topology on the set ListB we are considering the set of axioms $I(l) = \{*\}$ and $C(l, *) = \{l0, l1\}$, which considers a single axiom for any element l in ListB, and in the case of the formal topology on the set ListN we are considering the set of axioms $I(l) = \{*\}$ and $C(l, *) = \{lx \mid x \in \mathsf{N}\}$, which again considers a single axiom for any element l in ListN. To check that such axioms satisfy the requirement for an inductively generated formal cover one has only to check that, if $l, m \in$ ListB (respectively, $l, m \in$ ListN) and $l \sqsubseteq m$ then $\{l0, l1\} \subseteq \{w \in \mathsf{ListB} \mid (w \sqsubseteq l)\ \&\ (\exists z \in \{m0, m1\})\ w \sqsubseteq z\}$ (respectively, $\{lx \mid x \in \mathsf{N}\} \subseteq \{w \in \mathsf{ListN} \mid (w \sqsubseteq l)\ \&\ (\exists z \in \{mx \mid x \in \mathsf{N}\})\ w \sqsubseteq z\}$) which is trivial.

1.2. The bar theorem

One may wonder about the relation between the inductive definition of bar subset in the previous section and the original one that we recalled in the introductory section.

The first step to make possible such a comparison is expressing the notion of infinite sequence within our framework. One possibility is identifying an infinite sequence with a subset of lists of any possible length, one extending the other, namely, a subset α satisfying the following conditions (the conditions on the left apply to the case of the complete binary tree while the ones on the right apply to the case of the complete countable spreading tree):

	binary tree	**spread**
(non-emptiness)	$\mathsf{nil} \in \alpha$	$\mathsf{nil} \in \alpha$
(completeness)	$\dfrac{l \in \alpha}{l0 \in \alpha \vee l1 \in \alpha}$	$\dfrac{l \in \alpha}{(\exists x \in \mathsf{N})\ lx \in \alpha}$
(consistency)	$\dfrac{l0 \in \alpha \quad l1 \in \alpha}{\bot}$	$\dfrac{x \neq y \quad lx \in \alpha \quad ly \in \alpha}{\bot}$

Now, it is interesting to notice that inductively generated formal topologies provide a natural way to deal with the notion of infinite sequence (see for instance [14]).

Definition 1.2 (*Formal Point*). Let $\mathcal{A}$ be an inductively generated formal topology, with order relation $\leq$ and axiom set $I(-)$ and $C(-, -)$. Then, a non-empty subset α of A is a *formal point* if it satisfies the following conditions:

$$\text{(splitness)} \quad \frac{a \in \alpha \quad i \in I(a)}{(\exists y \in C(a, i))\ y \in \alpha}$$

$$\text{(directness)} \quad \frac{a \in \alpha \quad b \in \alpha}{(\exists c \in a \downarrow b)\ c \in \alpha}$$

where $a \downarrow b \equiv \{c \in A \mid (c \leq a)\ \&\ (c \leq b)\}$.

[1] It can be useful to observe that in the cited papers, instead of *extended reflexivity*, the rules of *reflexivity*, namely, $a \in U$ yields $a \lhd U$, and $\leq$*-left*, namely, $a \leq b$ and $b \lhd U$ yield $a \lhd U$, are considered. However, the first can be proved trivially from our *extended reflexivity* and the second can be obtained by induction on the length of a proof of $b \lhd U$ obtained by using only *extended reflexivity* and *infinity*.

Indeed, it is easy to check that the infinite sequences of the complete binary tree correspond to the formal points of the formal topology on ListB, and the infinite sequences of the complete countable spreading tree correspond to the formal points of the formal topology on ListN.

We can now immediately obtain the equivalence between Definition 1.1 of the inductive bar subset and the standard one as a consequence of the following general result on countably presented formal topologies (see [15]). In order to state it we have to explain what it means for an inductively generated formal topology to have a countable axiom set. To this end, let A be a set and $I(-)$, $C(-,-)$ be an axiom set for a formal topology over A. Then such an axiom set is countable if the set $\Sigma(A, I) \equiv \{\langle a, b\rangle \mid a \in A,\ b \in I(a)\}$ is countable.[2]

Theorem 1.3 (*Countable Completeness*). *Let $\mathcal{A}$ be an inductively generated formal topology with a countable axiom set. Then, for any $a \in A$ and $U \subseteq A$, $a \lhd U$ if and only if, for any formal point α, $a \in \alpha$ yields that there exists $u \in U$ such that $u \in \alpha$.*

Indeed, we know that U is a bar for the formal cover of the complete binary tree (respectively, for the formal cover of the complete countable spreading tree) if and only if $\mathsf{nil} \lhd U$ and hence, by the previous completeness theorem, if and only if, for any formal point α, $\mathsf{nil} \in \alpha$ yields that there exists $u \in U$ such that $u \in \alpha$; but any formal point contains the empty list nil and hence the previous condition is equivalent to the fact that, for any formal point α, there exists $u \in \alpha$ such that $u \in U$ which means that, for any infinite sequence α, there exists a finite initial segment u of α such that $u \in \overline{U}$, namely, α is barred by U.

It can be useful to observe that the completeness theorem above justifies the fact of considering only *extended reflexivity* and *infinity*, among all the possible valid conditions, when formalizing the notion of bar subset.

However, it is important to recall that the proof of the theorem of countable completeness requires the use of classical logic and hence the proof here is only useful to a classical reader which wants to understand why an intuitionistic friend is calling bar subset what such a friend obtains by using an inductive definition.

2. Characterizations of the inductive bar subsets

In order to gain a better comprehension of what an inductive bar subset is, we want to show now some equivalent formulations of this notion both for the binary tree and for the countable spreading tree.

2.1. The case of the binary tree

Since *extended reflexivity* and *infinity* are the inductive rules defining the cover relation, the subset $\lhd(U) \equiv \{a \in \mathsf{listB} \mid a \lhd U\}$ is the minimal solution of the following equation on the indeterminate X ranging over subsets of ListB:

$$l \in X \quad \text{iff} \quad (l \in \overline{U}) \vee (l0 \in X \ \&\ l1 \in X).$$

After the Tarski fixed-point theorem, it is immediate to see that such an equation has a solution since the associated operator is monotone, but we can also build a predicative solution by introducing the following chain of subsets:

$$\begin{array}{lll} U_0 & \equiv & \overline{U} \\ U_{n+1} & \equiv & U_n \cup \{l \in \mathsf{ListB} \mid (l0 \in U_n)\ \&\ (l1 \in U_n)\}. \end{array}$$

Indeed, the following lemma holds.

Lemma 2.1. *Let U be a subset of the complete binary tree. Then, $\lhd(U) = \bigcup_{n\in\mathsf{N}} U_n$.*

Proof. We have to show that $\bigcup_{n\in\mathsf{N}} U_n$ satisfies *extended reflexivity* and *infinity* and that any subset which contains $\overline{U}$ and enjoys *infinity* contains also $\bigcup_{n\in\mathsf{N}} U_n$.

[2] The condition of countability of the axiom set in the next theorem is required in order to carry on a proof by induction where it is necessary to take care of all the axioms that, for this reason, have to be enumerated by natural numbers; note that the proof requires classical reasoning and hence such an enumeration can be provided within ZFC with no loss of generality (see [15] for the detail of the proof and some comments on its interest from a constructive point of view).

Now, if $l \in \overline{U}$, i.e. $l \in U_0$, then $l \in \bigcup_{n \in \mathsf{N}} U_n$. Moreover, if $l0 \in \bigcup_{n \in \mathsf{N}} U_n$ and $l1 \in \bigcup_{n \in \mathsf{N}} U_n$ then there exist $n_0, n_1 \in \mathsf{N}$ such that $l0 \in U_{n_0}$ and $l1 \in U_{n_1}$; but the family of subsets U_n is a chain and hence $l \in U_{\max(n_0,n_1)+1}$; thus we conclude that $l \in \bigcup_{n \in \mathsf{N}} U_n$.

Suppose now that W is a subset which contains $\overline{U}$ and enjoys *infinity*. Then $U_0 = \overline{U} \subseteq W$ and, for any $n \in \mathsf{N}$, if $U_n \subseteq W$ then $U_{n+1} \subseteq W$ since $\{l \in \mathsf{ListB} \mid (l0 \in U_n) \,\&\, (l1 \in U_n)\} \subseteq W$, because W enjoys *infinity*. Thus $\bigcup_{n \in \mathsf{N}} U_n \subseteq W$. ■

2.1.1. A characterization of inductive bar subsets

In order to find an alternative formal description of what an inductive bar is, let us observe that, from an intuitive point of view a subset U is a bar if its elements are sufficient to 'block' every infinite path from the root. Guided by this idea, we are ready to introduce the first alternative definition of what a bar is for the complete binary tree.

Definition 2.2 (*Natural Bounded Bar Subset*). Let U be a subset of ListB. Then U is a *natural bounded bar subset* if and only if there exists a natural number $n \in \mathsf{N}$ such that, for all $l \in \mathsf{ListB}$, if $\mathsf{length}(l) \geq n$ then $l \in \overline{U}$.

The idea underlying this definition should be clear: if U has to be a bar subset then there must be some depth in the complete binary tree such that all the possible paths of length n already met U and hence they are within $\overline{U}$. Of course, from a constructive viewpoint this definition of bar subset is very strong, namely, it is going to be difficult to prove that a subset is a bar, because the constructive meaning of the existential quantifier forces us to say that U is a bar subset only when we can explicitly furnish a finite bound to the lengths of its elements.

However, it is possible to prove the following lemma.

Theorem 2.3. *Let U be a subset of the complete binary tree. Then U is a natural bounded bar subset if and only if it is an inductive bar subset.*

Proof. Let us suppose that U is a natural bounded bar subset. Then, for some $n \in \mathsf{N}$, $(\mathsf{length}(l) \geq n) \to (l \in \overline{U})$ holds for all $l \in \mathsf{ListB}$. Then, for any $l \in \mathsf{ListB}$ whose length is greater than or equal to n, $l \in U_0$, where U_0 is the first subset in the chain of subsets that we defined in the end of the previous section in order to prove Lemma 2.1. Now, if $n = 0$ we are done since we have proved that all the lists of length 0, namely, the single list nil, belongs to the subset U_0 and hence $\mathsf{nil} \lhd U$ holds by *extended reflexivity*. Otherwise, by definition, all the lists l of length $n-1$ are elements of U_1, since $l0$ and $l1$ belong to U_0 because all the lists of length n do, and hence, step after step, we will arrive at a proof that all the lists of length 0 are elements of U_n, namely, $\mathsf{nil} \in U_n$, and hence $\mathsf{nil} \lhd U$ follows by Lemma 2.1.

To prove the other implication, namely, if $\mathsf{nil} \lhd U$ then there exists $n \in \mathsf{N}$ such that any list of length greater than or equal to n belongs to $\overline{U}$, we will argue by induction on the length of the derivation of $a \lhd U$ in order to show a generalization of the required statement, namely, we will show that if $a \lhd U$ then there exists a natural number n such that, for any list l of length greater than or equal to n, $a + l \in \overline{U}$ where by $a + l$ we mean the concatenation of the lists a and l. Indeed, if $a \lhd U$ because $a \in \overline{U}$ then, for any list l, that is, for any list whose length is greater than or equal to 0, $a + l \in \overline{U}$. Moreover, if $a \lhd U$ because $a0 \lhd U$ and $a1 \lhd U$ then, by the inductive hypothesis, there are two natural numbers n_0 and n_1 such that, for all the lists l_0 whose length is at least n_0, $a0 + l_0 \in \overline{U}$ and, for all the lists l_1 whose length is at least n_1, $a1 + l_1 \in \overline{U}$; then, for all the lists m whose length is at least $\max(n_0, n_1) + 1$ the list $a + m$ belongs to $\overline{U}$. ■

2.1.2. Yet another characterization

After Theorem 2.3, Definition 2.2 of natural bounded bar subset is the only one (apart from logical equivalence) guaranteeing that a bar subset can be inductively generated as a formal cover. On the other hand, it can be useful to propose other definitions for a bar subset, even if they have to give rise to something which has to be logically equivalent to Definition 2.2. Indeed, such definitions can still be useful to gain a better comprehension of what an inductive bar is. For instance, in this section we will show an alternative characterization of the bar subsets for the complete binary tree which is more suitable for the extension to the complete countable spreading tree that we are going to deal with in the next section. The idea for such an alternative characterization is defining a suitable set which allows one to better approximate the 'border' of the bar subset than just a single natural number. That is why we introduce the set Ord_2 of the *binary ordinals*:

$$(\mathsf{Ord}_2\text{-int.}) \quad 0 \in \mathsf{Ord}_2 \qquad \frac{a_0 \in \mathsf{Ord}_2 \quad a_1 \in \mathsf{Ord}_2}{\langle a_0, a_1\rangle \in \mathsf{Ord}_2}.$$

Now, we can introduce a function $\mathsf{In}(l, o)$, which states whether a list l of boolean values is 'within' a binary ordinal o or has not 'entered' it yet, as follows:

$$\mathsf{In}(l, o) \equiv \begin{cases} \mathsf{true} & \text{if } o = 0 \\ \mathsf{false} & \text{if } o = \langle o_0, o_1\rangle \text{ and } l = \mathsf{nil} \\ \mathsf{In}(m, o_0) & \text{if } o = \langle o_0, o_1\rangle \text{ and } l = 0m \\ \mathsf{In}(m, o_1) & \text{if } o = \langle o_0, o_1\rangle \text{ and } l = 1m \end{cases}$$

and thus we arrive at the following definition.

Definition 2.4 (*Binary Ordinal Bounded Bar Subset*). Let U be a subset of the complete binary tree. Then U is a *binary ordinal bounded bar subset* if and only if there exists a binary ordinal $o \in \mathsf{Ord}_2$ such that, for all $l \in \mathsf{ListB}$, if $\mathsf{In}(l, o)$ then $l \in \overline{U}$.

It is not difficult to prove that a subset of lists is naturally bounded if and only if it is binary ordinal bounded and hence, after Theorem 2.3, both of these definitions are equivalent to that of the inductive bar subset. Indeed, if U is a binary ordinal bounded bar subset we can use the following map:

$$\mathsf{depth}(o) \equiv \begin{cases} 0 & \text{if } o = 0 \\ \mathsf{max}(\mathsf{depth}(o_1), \mathsf{depth}(o_2)) + 1 & \text{if } o = \langle o_0, o_1\rangle \end{cases}$$

to find, given a binary ordinal o, a natural number $\mathsf{depth}(o)$ such that $\mathsf{length}(l) \geq \mathsf{depth}(o)$ yields $\mathsf{In}(l, o)$, and, on the other hand, if U is a natural bounded bar subset we can use the following map:

$$\mathsf{nat2binord}(n) \equiv \begin{cases} 0 & \text{if } n = 0 \\ \langle \mathsf{nat2binord}(m), \mathsf{nat2binord}(m)\rangle & \text{if } n = m + 1 \end{cases}$$

to find, given a natural number n, a binary ordinal $\mathsf{nat2binord}(n)$ such that $\mathsf{In}(l, \mathsf{nat2binord}(n))$ yields $\mathsf{length}(l) \geq n$.

2.2. *The case of the spread*

Also in the case of a spread we can develop a proof for a lemma which is analogous to Lemma 2.1, but in order to obtain such a result we cannot limit ourselves to induction on natural numbers but have to use induction over the set Ord of the ordinals (see [8]):

$$(\mathsf{Ord}\text{-int.}) \quad 0 \in \mathsf{Ord} \qquad \frac{a \in \mathsf{Ord}}{a + 1 \in \mathsf{Ord}} \qquad \frac{h \in \mathsf{N} \to \mathsf{Ord}}{\Lambda(h) \in \mathsf{Ord}}.$$

Indeed, given any $U \subseteq \mathsf{ListN}$, by using such a set, we can provide a constructive solution of the following subset equation on the indeterminate X:

$$l \in X \quad \text{iff} \quad (l \in \overline{U}) \vee ((\forall x \in \mathsf{N})\, lx \in X)$$

since we can define the following family of subsets by induction on the set Ord:

$$\begin{array}{lcl} U_0 & \equiv & \overline{U} \\ U_{n+1} & \equiv & U_n \cup \{l \in \mathsf{ListN} \mid (\forall x \in \mathsf{N})\, lx \in U_n\} \\ U_{\Lambda(h)} & \equiv & \bigcup_{n \in \mathsf{N}} U_{h(n)} \qquad \text{if } h : \mathsf{N} \to \mathsf{Ord}. \end{array}$$

Then, we can prove the following lemma whose statement is analogous to that of Lemma 2.1. Also its proof is very similar to that of such a lemma, but we think that it is worth repeating it here since now there is an essential use of the *axiom of choice* (see [7]) that was not necessary in the previous case. It can be useful to note that the use of the axiom of choice is here completely justified since we are working with inductive sets and hence the axiom of choice is just a consequence of the constructive meaning of the disjoint sum constructor.

Lemma 2.5. *Let U be a subset of the complete countable spreading tree. Then,* $\lhd(U) = \bigcup_{o \in \mathsf{Ord}} U_o$.

Proof. We have to show that $\bigcup_{o \in \mathsf{Ord}} U_o$ satisfies *extended reflexivity* and *infinity* and that any subset which contains $\overline{U}$ and enjoys *infinity* contains $\bigcup_{o \in \mathsf{Ord}} U_o$.

Now, if $l \in \overline{U}$, i.e., $l \in U_0$, then $l \in \bigcup_{o \in \mathsf{Ord}} U_o$. Moreover, if, for all $x \in \mathsf{N}$, $lx \in \bigcup_{o \in \mathsf{Ord}} U_o$ then, for each $x \in \mathsf{N}$, there exists $o_x \in \mathsf{Ord}$ such that $lx \in U_{o_x}$; then, by the axiom of choice, there exists a function $h : \mathsf{N} \to \mathsf{Ord}$ such that, for any $x \in \mathsf{N}$, $lx \in U_{h(x)}$; thus, $lx \in U_{\Lambda(h)}$, since $U_{h(x)} \subseteq U_{\Lambda(h)}$, and hence $l \in U_{\Lambda(h)+1}$; so $l \in \bigcup_{o \in \mathsf{Ord}} U_o$ holds.

Suppose now that W is a subset which contains $\overline{U}$ and enjoys *infinity*. Then $U_0 = \overline{U} \subseteq W$ and, for any $o \in \mathsf{Ord}$, if $U_o \subseteq W$ then $U_{o+1} \subseteq W$ since $\{l \in \mathsf{ListN} \mid (\forall x \in \mathsf{N})\ lx \in U_o)\} \subseteq W$, because W enjoys *infinity*. Moreover, if $h : \mathsf{N} \to \mathsf{Ord}$ and, for all $n \in \mathsf{N}$, $U_{h(n)} \subseteq W$, then $U_{\Lambda(h)} = \bigcup_{n \in \mathsf{N}} U_{h(n)} \subseteq W$. Thus $\bigcup_{o \in \mathsf{Ord}} U_o \subseteq W$. ■

Let us face now the problem of finding alternative characterizations of the notion of bar subset for a spread. Here we cannot just adapt to the new setting the previous Definition 2.2 which relies on the existence of a single natural number since it is well possible that, even if any path from the root is barred by U, there is no natural number which limits the number of steps which are necessary in order to arrive at such a bar; indeed, each node has a countable number of successors and all of them can require a different and increasing number of steps. However, we can generalize to the present setting the characterization of a bar subset that we proposed in Section 2.1.2. Indeed, a subset U is a bar for a spread if there is a limit to the length of a path in any possible 'direction' before it meets U. This is why we introduce now the set Ord^* of the star-ordinals, that is, a modified version of the ordinals where we consider only the ordinals which are countable spreading:

$$(\mathsf{Ord}^*\text{-int.}) \quad 0^* \in \mathsf{Ord}^* \qquad \frac{h \in \mathsf{N} \to \mathsf{Ord}^*}{\Lambda^*(h) \in \mathsf{Ord}^*}.$$

Now, we can adapt to the new setting the definition of the function $\mathsf{In}(-,-)$ that we introduced in Section 2.1.2:

$$\mathsf{In}(l, o) \equiv \begin{cases} \mathsf{true} & \text{if } o = 0^* \\ \mathsf{false} & \text{if } o = \Lambda^*(h) \text{ and } l = \mathsf{nil} \\ \mathsf{In}(l_1, h(x)) & \text{if } o = \Lambda^*(h) \text{ and } l = xl_1 \end{cases}$$

where by $l = xl_1$ we mean that the non-null list l begins with the number x and continues with the list l_1, and thus we arrive at the following definition.

Definition 2.6 (*Countable Ordinal Bar Subset*). Let U be a subset of the complete countable spreading tree. Then U is a *countable ordinal bar subset* if and only if there exists a star-ordinal $o \in \mathsf{Ord}^*$ such that, for all $l \in \mathsf{ListN}$, if $\mathsf{In}(l, o)$ then $l \in \overline{U}$.

Now we want to prove a result analogous to Theorem 2.3; we will do it in the next two theorems.

Theorem 2.7. *Let U be a subset of the complete countable spreading tree. Then if U is a countable bar subset, it is an inductive bar subset.*

Proof. We have to show that if there exists a star-ordinal o such that $l \in \overline{U}$ holds for all lists l such that $\mathsf{In}(l, o)$ then $\mathsf{nil} \lhd U$. We will prove this result by induction on the complexity of the star-ordinal o.

To this end, given $x \in \mathsf{N}$, let us describe as U_x the subset $\{l \in \mathsf{ListN} \mid xl \in \overline{U}\}$, where by xl we mean a list which begins with the number x.

Now, if the star-ordinal o is equal to 0^* then, for all the lists l such that $\mathsf{In}(l, 0^*)$ holds, namely, for all the lists, $l \in \overline{U}$ and hence, in particular, $\mathsf{nil} \in \overline{U}$ and thus $\mathsf{nil} \lhd U$ follows by *extended reflexivity*.

On the other hand, namely, if $o \equiv \Lambda^*(h)$, then it is immediate that, for any $x \in \mathsf{N}$ and any $l \in \mathsf{ListN}$, if $\mathsf{In}(l, h(x))$ then $l \in \overline{U_x}$; indeed, if $\mathsf{In}(l, h(x))$ then $\mathsf{In}(xl, \Lambda^*(h))$ and hence, by assumption, $xl \in \overline{U}$ which yields that $l \in U_x$ and so $l \in \overline{U_x}$. Then, $\mathsf{nil} \lhd U_x$ follows by the inductive hypothesis since, for any $x \in \mathsf{N}$, the star-ordinal $h(x)$ is smaller than $\Lambda^*(h)$.

Let us prove now, by induction on the length of the derivation, that if $l \lhd U_x$ then $xl \lhd U$. Consider first the case where $l \lhd U_x$ has been obtained by *extended reflexivity*, namely, there exist $u \in U_x$ such that $l \sqsubseteq u$; then $xl \sqsubseteq xu$, that is, there exists $v \in \overline{U}$ such that $xl \sqsubseteq v$; so also $xl \in \overline{U}$ and hence $xl \lhd U$ follows by *extended reflexivity*. On the other hand, if $l \lhd U_x$ has been obtained by *infinity* then, for any $y \in \mathsf{N}$, $ly \lhd U_x$ and so, by the inductive hypothesis, $xly \lhd U$; hence, we get $xl \lhd U$ by *infinity*.

Thus, $\mathsf{nil} \lhd U_x$ yields $\mathsf{nil}\, x \lhd U$ and we can finally conclude $\mathsf{nil} \lhd U$ by *infinity*. ■

We can also prove the other implication. Here again the proof technique is similar to the one that we used in the case of the binary tree apart for the use of the axiom of choice.

Theorem 2.8. *Let U be a subset of the complete countable spreading tree. Then if U is an inductive bar subset, it is a countable ordinal bar subset.*

Proof. We have to prove that if $\mathsf{nil} \lhd U$ then there exists a star-ordinal o such that, for any list l such that $\mathsf{In}(l, o)$, $l \in \overline{U}$. We will prove this result by showing by induction on the length of the proof of the proposition $a \lhd U$ that there exists a star-ordinal o such that, for any list l such that $\mathsf{In}(l, o)$, $a + l \in \overline{U}$.

So, let us suppose that $a \lhd U$ because $a \in \overline{U}$; then, for any list l, $a + l \in \overline{U}$ and hence we can get the required result by using the star-ordinal 0^*.

On the other hand, if $a \lhd U$ has been obtained from $(\forall x \in \mathsf{N})\ ax \lhd U$ then, by the inductive hypothesis, we get that for all $x \in \mathsf{N}$ there exists a star-ordinal o_x such that, for any list l such that $\mathsf{In}(l, o_x)$, $ax + l \in \overline{U}$; then, by the axiom of choice, there exists a function h from the set N to the set Ord^* such that for all $x \in \mathsf{N}$ and all $l \in \mathsf{ListN}$, if $\mathsf{In}(l, h(x))$ then $ax + l \in \overline{U}$. But $\mathsf{In}(xl, \Lambda^*(h))$ if and only if $\mathsf{In}(l, h(x))$ and hence we found a star-ordinal, namely, $\Lambda^*(h)$, such that for any list xl, if $\mathsf{In}(xl, \Lambda^*(h))$ then $a + xl \equiv ax + l \in \overline{U}$. ■

It is worth noting that all of what we proved in this section for a countable spreading tree can be trivially generalized to a tree which is X-spreading, namely, a tree such that the immediate successors of each node are as many as the elements of a set X. Indeed, to obtain such a result it is sufficient to replace in all the statements in this section the set N with the set X, namely, considering the set $\mathsf{List}X$ of the lists of elements of X instead of the set ListN, the set Ord_X of the X-spreading ordinals whose introduction rules are

$$(\mathsf{Ord}_X\text{-int.}) \quad 0 \in \mathsf{Ord}_X \qquad \frac{a \in \mathsf{Ord}_X}{a + 1 \in \mathsf{Ord}_X} \qquad \frac{h \in X \to \mathsf{Ord}_X}{\Lambda(h) \in \mathsf{Ord}_X}$$

instead of the set Ord and the set Ord^*_X, obtained by considering only the first and the last rules for Ord_X, instead of the set Ord^*.

2.3. *Bar subsets for formal topologies*

In the previous sections we studied the notions of bar subset for the complete binary tree and for the countable spreading tree. But, we also observed that these are just special cases of the more abstract notion of inductively generated formal topology. Thus, it is natural to wonder whether there exists a notion of bar subset for an inductively generated formal topology and how to characterize such a notion. Even if in this case we cannot rely on a nil element, and hence we cannot express the notion of inductive bar by stating that U is an inductive bar subset if and only if $\mathsf{nil} \lhd U$, it is not difficult to find an equivalent formulation that can as well be used in the general case. In fact, it is almost immediate that, for any subset U of ListB or ListN, $\mathsf{nil} \lhd U$ if and only if $l \lhd U$ holds for any list l. Indeed, one direction is trivial while to obtain the other one we can prove by induction that, for any list $l \in \mathsf{ListB}$ (or $l \in \mathsf{ListN}$) and for any $x \in \mathsf{Boole}$ (or $x \in \mathsf{N}$), if $l \lhd U$ then $lx \lhd U$. Indeed, if $l \lhd U$ has been obtained by *extended reflexivity* then $l \in \overline{U}$ and hence also $lx \in \overline{U}$, so $lx \lhd U$ follows by *extended reflexivity*; moreover, if $l \lhd U$ has been obtained by *infinity* then, for any $y \in \mathsf{Boole}$ (or $y \in \mathsf{N}$), $ly \lhd U$ and hence we get $lx \lhd U$ by logic.

Thus we arrive at the following definition.

Definition 2.9 (*Inductive Bar for a Formal Topology*). Let $\mathcal{A}$ be an inductively generated formal topology and U be a subset of A. Then U is an *inductive bar subset for the formal topology* $\mathcal{A}$ if and only if, for any $a \in A$, $a \lhd U$.

After this definition, the first step to arrive at some alternative characterization of bar subsets is finding the correct formulation for a result similar to Lemmas 2.1 or 2.5. To this end, we will introduce a slightly modified version of the axioms to generate by induction a formal topology. Our purpose is finding suitable axioms making it possible to show an explicit solution of the inductive definition of the cover relation, namely, to find the minimal subset X of A such that, for any $a \in A$,

$$a \in X \quad \text{if and only if} \quad (a \in \overline{U}) \vee ((\exists i \in I(a))(\forall y \in C(a,i))\ y \in X).$$

Thus, let us consider like a set of axioms a triple $I(-)$, $D(-,-)$ and $d(-,-,-)$ such that $I(a)$ set, for any $a \in A$, $D(a,i)$ set, for any $a \in A$ and $i \in I(a)$, and $d(a,i,x) \in A$ for any $a \in A$, $i \in I(a)$ and $x \in D(a,i)$. The intended meaning of such new axioms is that, for all $a \in A$ and $i \in I(a)$, a is covered by the subset $\mathsf{Im}[d(a,i)] \equiv \{d(a,i,x) \mid x \in D(a,i)\}$.

Note that, provided we have a "new" set of axioms, namely, a triple I, D and d, it is trivial to obtain an "old" set of axioms, namely, a couple I and C, by simply considering the same family of sets $I(a)$ and setting

$$C(a,i) \quad \equiv \quad \mathsf{Im}[d(a,i)] \quad \text{for any } a \in A \text{ and } i \in I(a).$$

It is also possible to go in the other direction (see the last section of [3] or [5]). Indeed, supposing I and C is a set of "old" axioms, we can obtain a set of "new" axioms by considering the same family of sets $I(a)$ and stating

$$\begin{array}{rcll} D(a,i) & \equiv & \Sigma(A, C(a,i)) & \text{for any } a \in A \text{ and } i \in I(a) \\ d(a,i,x) & \equiv & \mathsf{fst}(x) & \text{for any } a \in A,\ i \in I(a) \text{ and } x \in D(a,i) \end{array}$$

where $\Sigma(-,-)$ is the set constructor for the disjoint sum of a family of sets (see [7]).

Let us show now why this simple change in the shape of the axioms is sufficient to solve the inductive definitions of the cover relation.

First of all, let us rewrite the rules defining the cover relation of an inductively generated formal topology in this new formal setting:

$$\text{(ext.-reflexivity)} \quad \frac{a \in \overline{U}}{a \lhd U} \qquad \text{(infinity)} \quad \frac{i \in I(a) \quad (\forall x \in D(a,i))\ d(a,i,x) \lhd U}{a \lhd U}.$$

Then, let us recall the game theoretic approach to inductive definition [1]. Coquand in [2] used this game analogy to give an inductive definition of a coinductively defined predicate and T. Coquand and P. Martin-Löf used the same approach to present a cover relation (see [12]). Let us suppose that I, D and d is a set of axioms to generate inductively a formal topology on a set A that we are going to consider like the states of a game. Then we can introduce the following game between two players P and Q. Suppose that U is a subset of A and that P is in turn of moving in the state $a \in A$. Then a move for P consists in choosing any element $i \in I(a)$ and, after such a move, a move for Q consists in choosing any element $x \in D(a,i)$ in order to arrive at the new state $d(a,i,x)$. Now, the first player P wins the game if he can force Q to choose an element in the subset $\overline{U}$ while Q wins if he is able to stay always "out" of $\overline{U}$. Then, the position a is a winning position for the player P if and only if $a \in \overline{U}$ or there exists $i \in I(a)$ such that, for all $x \in D(a,i)$, $d(a,i,x)$ is a winning position for P. So, due to the equation defining the cover relation, there is a winning strategy for P if and only if the element a is covered by the subset U.

If we want to find an explicit solution for this game we can try to define the cover relation by using the paths on the game tree that we obtain (a similar construction can be found in [9] but there the set of the well ordering is used while here we use an instance of the set of the tree constructor [10]). First of all let us note that, as in the case of a spread, there is no predefined bound for the length of a possible winning path for P. Thus, we cannot limit ourselves to any natural number of steps but we have to adapt the set of the ordinals that we used in the previous section to our new setting. This is why we introduce the following family of sets, namely, the *ordinals over A*:

$$\frac{a \in A}{0_a \in \mathsf{Ord}_A(a)}$$

$$\frac{n \in \mathsf{Ord}_A(a)}{n+1 \in \mathsf{Ord}_A(a)}$$

$$\frac{a \in A \quad i \in I(a) \quad f \in (\forall x \in D(a,i))\ \mathsf{Ord}_A(d(a,i,x))}{\Lambda(a,i,f) \in \mathsf{Ord}_A(a)}.$$

Suppose now that U is any subset of A and define the following family of subsets of A indexed, for any $a \in A$, on the elements of $\mathsf{Ord}_A(a)$:

$$\begin{array}{lcl} U_{0_a} & = & \overline{U} \\ U_{n+1} & = & U_n \cup \{c \in A \mid (\exists i \in I(c))(\forall x \in D(c,i))\, d(c,i,x) \in U_n\} \\ U_{\Lambda(a,i,f)} & = & \bigcup_{x \in D(a,i)} U_{f(x)} \end{array}$$

where the last clause applies when $i \in I(a)$ and $f \in (\forall x \in D(a,i))\, \mathsf{Ord}_A(d(a,i,x))$.

Now, let us state that $a \lhd U$ holds if and only if there exists $o \in \mathsf{Ord}_A(a)$ such that $a \in U_o$. Then it is possible to prove that the cover relation defined in such a way enjoys both *extended reflexivity* and *infinity*.

- (extended reflexivity) We have to prove that if $a \in \overline{U}$ then $a \lhd U$. This result is immediate; indeed if $a \in \overline{U}$ then $a \in U_{0_a}$ and hence there exists $o \in \mathsf{Ord}_A(a)$ such that $a \in U_o$.
- (infinity) We have to prove that, if there exists $i \in I(a)$ such that for all $x \in D(a,i)$, $d(a,i,x) \lhd U$ then $a \lhd U$. So, let us assume that for any $x \in D(a,i)$, $d(a,i,x) \lhd U$. Then, by definition, there exists $o_x \in \mathsf{Ord}_A(d(a,i,x))$ such that $d(a,i,x) \in U_{o_x}$. Thus, by the axiom of choice, there exists a function $f \in (\forall x \in D(a,i))\, \mathsf{Ord}_A(d(a,i,x))$ such that, for any $x \in D(a,i)$, $d(a,i,x) \in U_{f(x)}$. Hence $d(a,i,x) \in U_{\Lambda(a,i,f)}$ since, for any $x \in D(a,i)$, $U_{f(x)} \subseteq U_{\Lambda(a,i,f)}$. So, for any $x \in D(a,i)$, $d(a,i,x) \in U_{\Lambda(a,i,f)}$ and hence, $a \in U_{\Lambda(a,i,f)+1}$.

Moreover, $\lhd(U)$ is the smallest subset of A such that both *extended reflexivity* and *infinity* hold. We can prove this result by showing that, provided V is a subset of A such that $\overline{U} \subseteq V$ and $\mathsf{Im}[d(c,j)] \subseteq V$ yields $c \in V$, then, for any $a \in A$, if $a \lhd U$ then $a \in V$. Since $a \lhd U$ means that there exists an ordinal $o \in \mathsf{Ord}_A(a)$ such that $a \in U_o$, the proof is by induction on the construction of the set $\mathsf{Ord}_A(a)$. Now, $U_{0_a} = \overline{U} \subseteq V$ holds by *extended reflexivity* for V, and if $U_n \subseteq V$ then also $U_{n+1} \subseteq V$ since, by *infinity* for V, if for some $c \in A$ there is $i \in I(c)$ such that, for all $x \in D(c,i)$, $d(c,i,x) \in V$, then $c \in V$. Finally, if, for some $i \in I(a)$ and $f \in (\forall x \in D(a,i))\, \mathsf{Ord}_A(d(a,i,x))$, it happens that, for all $x \in D(a,i)$, $U_{f(x)} \subseteq V$ then $U_{\Lambda(a,i,f)} = \bigcup_{x \in D(a,i)} U_{f(x)} \subseteq V$.

Also in the case of an inductively generated formal topology we can provide an alternative characterization of the notion of bar subset. To this end, in a way completely similar to what we did in the previous sections, we will introduce the star-ordinal over A by using the following rules:

$$\frac{a \in A}{0^*_a \in \mathsf{Ord}^*_A(a)} \qquad \frac{a \in A \quad i \in I(a) \quad f \in (\forall x \in D(a,i))\, \mathsf{Ord}^*_A(d(a,i,x))}{\Lambda^*(a,i,f) \in \mathsf{Ord}^*_A(a)}.$$

Then, given an element $a \in A$ and a star-ordinal $o^* \in \mathsf{Ord}^*_A(a)$, we can define the subset of the elements of A which cover a after o^* steps by setting

$$\mathsf{Final}(a,o^*) \equiv \begin{cases} \{a\} & \text{if } o^* = 0^*_a \\ \bigcup_{x \in D(a,i)} \mathsf{Final}(d(a,i,x), f(x)) & \text{if } o^* = \Lambda^*(a,i,f) \text{ for some} \\ & i \in I(a) \text{ and} \\ & f \in (\forall x \in D(a,i))\, \mathsf{Ord}^*_A(d(a,i,x)). \end{cases}$$

It is now possible to introduce the following definition.

Definition 2.10 (*Ordinal Bar Subset for a Formal Topology*). Let $\mathcal{A}$ be an inductively generated formal topology and U be a subset of A. Then U is an *ordinal bar subset for the formal topology* $\mathcal{A}$ if for any $a \in A$ there exists a star-ordinal $o^* \in \mathsf{Ord}^*_A(a)$ such that $\mathsf{Final}(a,o^*) \subseteq \overline{U}$.

The meaning of this definition should be clear: $\mathsf{Final}(a,o^*)$ contains elements that cover a by axiom after o^* steps and hence a is covered by U if there is an ordinal such that all such elements are in $\overline{U}$.

Thus it should not be surprising that it is possible to prove the following theorem. Note that this theorem is a specialization of theorem 4.4 in [3] where an instance of the *tree set*, corresponding to our type $\mathsf{Ord}^*_A(-)$, is used.

Theorem 2.11. *Let $\mathcal{A}$ be an inductively generated formal topology and U be a subset of A. Then U is an ordinal bar subset for the formal topology $\mathcal{A}$ if and only if it is an inductive bar subset.*

Proof. We have to show that, for any $a \in A$, there exists a star-ordinal $o^* \in \mathsf{Ord}^*_A(a)$ such that $\mathsf{Final}(a, o^*) \subseteq \overline{U}$ if and only if there exists an ordinal $o \in \mathsf{Ord}_A(a)$ such that $a \in U_o$.

To prove the left-to-right implication, let us first suppose $o^* = 0^*_a$; then $\mathsf{Final}(a, o^*) = \{a\}$ and hence $\mathsf{Final}(a, o^*) \subseteq \overline{U}$ means that $a \in \overline{U} = U_0$ and so we are done by setting $o = 0_a$. Moreover, if $o^* = \lambda^*(a, i, f)$ for some $i \in I(a)$ and $f \in (\forall x \in D(a, i))\ \mathsf{Ord}^*_A(d(a, i, x))$ then $\mathsf{Final}(a, o^*)$ is equal to $\bigcup_{x \in D(a,i)} \mathsf{Final}(d(a, i, x), f(x))$ and hence $\mathsf{Final}(a, o^*) \subseteq \overline{U}$ means that, for any $x \in D(a, i)$, $\mathsf{Final}(d(a, i, x), f(x)) \subseteq \overline{U}$; so, by the inductive hypothesis, for any $x \in D(a, i)$, there exists an ordinal $o_x \in \mathsf{Ord}_A(d(a, i, x))$ over A such that $d(a, i, x) \in U_{o_x}$ and hence, by the axiom of choice, there exist a function $h \in (\forall x \in D(a, i))\ \mathsf{Ord}_A(d(a, i, x))$ such that, for all $x \in D(a, i)$, $d(a, i, x) \in U_{h(x)}$ and thus $a \in U_{\lambda(a,i,h)+1}$.

To prove the other implication, let us first suppose that $o = 0_a$, that is, $a \in U_{0_a} \equiv \overline{U}$; then, it is sufficient to set $o^* \equiv 0^*_a$ in order to get $\mathsf{Final}(a, o^*) = \{a\} \subseteq \overline{U}$. Moreover, if $o = o'+1$ then $a \in U_{o'+1}$ means that either $a \in U_{o'}$, and hence the result follows by the inductive hypothesis, or there exists $i \in I(a)$ such that, for all $x \in D(a, i)$, $d(a, i, x) \in U_{o'}$; in this last case, by the inductive hypothesis, we get that there exists $i \in I(a)$ such that for all $x \in D(a, i)$ there exists a star-ordinal $o^*_x \in \mathsf{Ord}^*_A(d(a, i, x))$ such that $\mathsf{Final}(d(a, i, x), o^*_x) \subseteq \overline{U}$; thus, by the axiom of choice, there exists a function $f \in (\forall x \in D(a, i))\ \mathsf{Ord}^*_A(d(a, i, x))$ such that, for all $x \in D(a, i)$, $\mathsf{Final}(d(a, i, x), f(x)) \subseteq \overline{U}$; so, if we set $o^* \equiv \lambda^*(a, i, f)$ we get that $\mathsf{Final}(a, o^*) = \bigcup_{x \in D(a,i)} \mathsf{Final}(d(a, i, x), f(x)) \subseteq \overline{U}$. Finally, if $o = \lambda(a, i, h)$ for some $i \in I(a)$ and $h \in (\forall x \in D(a, i))\ \mathsf{Ord}_A(d(a, i, x))$ then $a \in U_o$ means that there exists some element $x \in D(a, i)$ such that $a \in U_{h(x)}$ and hence the result follows by the inductive hypothesis. ∎

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